GOOD REPRESENTATIONS AND SOLVABLE GROUPS

DAN EDIDIN AND WILLIAM GRAHAM

Dedicated to William Fulton on his 60th birthday

1. Introduction

The purpose of this paper is to provide a characterization of solvable linear algebraic groups in terms of a geometric property of representations. Representations with a related property played an important role in the proof of the equivariant Riemann-Roch theorem [EG2]. In that paper, we constructed representations with that property (which we call freely good) for the group of upper triangular matrices in GL_n. We noted that it seemed unlikely that such representations exist for arbitrary groups; the main result of this paper implies that they do not.

To state our results, we need some definitions. A representation V of a linear algebraic group G is said to be *good* (resp. *freely good*) if there exists a non-empty G-invariant open subset $U \subset V$ such that

(i) G acts properly (resp. freely) on U.

(ii) $V \setminus U$ is the union of a finite number of G-invariant linear subspaces.

Note that freely good representations were called good in [EG2].

The main result of the paper is the following theorem.

Theorem 1.1. *Let G be a connected algebraic group over a field k of characteristic not equal to 2. Then G is solvable if and only if G has a good representation. Moreover, if G is solvable and k is perfect then G has a freely good representation.*

In characteristic 2, a solvable group still has good representations, and a partial converse holds (Corollary 4.1). A key step in the proof of the main result is Theorem 4.1, which is inspired by an example of Mumford [GIT, Example 0.4].

In characteristic 0 solvable groups are characterized by a weaker property which does not require the action to be proper. (In general, if G acts properly on X, then G acts with finite stabilizers on X, but the converse need not hold.)

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Theorem 1.2. *Let G be a connected algebraic group over a field of characteristic* 0*. Suppose that G has a representation V that contains a nonempty open set U such that*

(1) *The complement of U is a finite union of invariant linear subspaces, and*

(2) *G acts with finite stabilizers on U.*

Then G is solvable.

Examples (see Section 6) show that this weaker property does not characterize solvability in positive characteristic.

2. Preliminaries

Groups and representations We let k denote a field, with algebraic closure $\overline{k}$ and separable closure k_s. If Z is a k-variety and $k' \supset k$ is any extension of k then $Z(k')$ denotes the k'-valued points of Z, while $Z_{k'}$ denotes the k'-variety $Z \times_k k'$.

All groups in this paper are assumed to be linear algebraic groups over a field k. We assume that such a group G is geometrically reduced, that is, that $G_{\overline{k}}$ is reduced. The identity component of a group G is denoted G^0.

Unless otherwise stated, a representation V of a group G is assumed to be k-rational; i.e. V is a k-vector space and the action map $G \times V \to V$ is a morphism of k-varieties.

If $k' \supset k$ is a field extension we call a k'-rational representation V of $G_{k'}$ a k'-representation of G. We say that V is defined over k if it is obtained by base change from a k-rational representation.

If $k' \supset k$ be a Galois field extension, then $\mathrm{Gal}(k'/k)$ acts on k'-representations of G. Indeed, let V be a k'-representation of G corresponding to a k' morphism $\rho\colon G_{k'} \to GL(V)$. For $g \in G(k_s)$ ${}^{\sigma}\rho(g)$ is defined as follows (cf. [Borel, AG14.3, 24.5]). Because ρ is defined over k', for any $\tau \in \mathrm{Gal}(k_s/k')$ and any $g \in G(k_s)$, $\tau(\rho(\tau^{-1}(g))) = \rho(g)$. Thus, if $\sigma \in \mathrm{Gal}(k'/k) = \mathrm{Gal}(k_s/k)/\mathrm{Gal}(k'/k)$, then

$$\sigma'(\rho((\sigma')^{-1}g)) \in GL(V)(k_s)$$

is independent of the lift of σ to an element $\sigma' \in \mathrm{Gal}(k_s/k)$. We will call this point $\sigma\rho(\sigma^{-1}(g))$ and set $\rho(g) = \sigma(\rho(\sigma^{-1}g))$.

The k'-representation V is obtained by base change from a representation defined over k if and only if ${}^{\sigma}\rho = \rho$ for all $\sigma \in \mathrm{Gal}(k'/k)$.

Free and proper actions The action of a group G on a scheme X is said to be *free* if the action map $G \times X \to X \times X$ is a closed embedding. The action is said to be *proper* if the map $G \times X \to X \times X$ is proper. If the action is proper then the stabilizer of every point is finite. If

the stabilizer of every geometric point is a trivial group-scheme then we say that the action is *set theoretically free*. An action which is set theoretically free and proper is free [EG1].

Let $H \to G$ be a finite morphism of algebraic groups. If G acts properly on a scheme X then H also acts properly on X. Thus, if V is a good representation of G then V is also a good representation of H via the action induced by the map $H \to G$. Moreover, if H is a closed subgroup and V is a freely good representation of G, then V is a freely good representation of H.

Example 2.1. Let B be the group of upper triangular matrices in $\mathrm{GL}(n)$. The group B acts by left multiplication on the vector space V of upper triangular matrices; it acts with trivial stabilizers on the open subset U of invertible upper triangular matrices. Since the matrices are upper triangular, $V \smallsetminus U$ is the union of the invariant subspaces $L_i = \{A \in V | A_{ii} = 0\}$. This representation is freely good because the action of B on U is identified with B acting itself by left multiplication. The map $B \times B \to B \times B$ given by $(A, A') \mapsto (A, AA')$ is an isomorphism, so the action of B on U is free.

By contrast, the action of $\mathrm{GL}(n)$ by left multiplication on the vector space M_n of $n \times n$ matrices is not good.

3. Existence of good representations

In this section we show that every connected solvable group has good representations, and if k is perfect, freely good representations.

By the Lie-Kolchin theorem $G_{\overline{k}}$ is trigonalizable; i.e., it can be embedded in the group $B_{\overline{k}} \subset \mathrm{GL}_n$ of upper triangular matrices.

Let $V_{\overline{k}}$ be the vector space of upper triangular $n \times n$ matrices. The group $B_{\overline{k}}$ acts on $V_{\overline{k}}$ by left multiplication and we have seen that this representation is freely good. By restriction $V_{\overline{k}}$ is a good representation of $G_{\overline{k}}$. Consider the morphism $\rho\colon G_{\overline{k}} \to \mathrm{GL}(V_{\overline{k}})$ corresponding to the action of $G_{\overline{k}}$ on $V_{\overline{k}}$.

Since ρ is a morphism of schemes of finite type, it is defined over a field extension $k' \supset k$ of finite degree. Write $V = V_{k'}$ for the corresponding k'-representation; then we have $\rho : G_{k'} \to \mathrm{GL}(V)$.

Case I. k' is separable over k. (This will occur when k is perfect.) In this case we will use Galois descent to construct a freely good representation of G.

Replacing k' by a possibly bigger field extension we may assume that $k' \supset k$ is Galois. Enumerate the elements of $\mathrm{Gal}(k'/k)$ as $\{1 = \sigma_1, \sigma_2, \dots \sigma_d\}$ and consider the representation $\Phi\colon G_{k'} \to \mathrm{GL}(V^{\oplus d})$ where $G_{k'}$ acts on the j-th factor by the representation ${}^{\sigma_j}\rho\colon G_{k'} \to \mathrm{GL}(V)$.

We define $U_d \subset V^{\oplus d}$ to be the open set whose k_s-rational points are the d-tuples $(A_1, \dots, A_d)$ where some A_i is invertible. We realize U_d as a complement of $G_{k'}$-invariant linear subspaces as follows. Let $L_j = \{A \in V | A_{jj} = 0\}$, a $G_{k'}$-invariant subspace of V. Given a d-tuple $(j_1, \dots, j_d)$, define

$$L_{(j_1,\dots,j_d)} = L_{j_1} \oplus \cdots \oplus L_{j_d}.$$

This is a $G_{k'}$-invariant subspace of $V^{\oplus d}$ and $U_d = V_d \setminus \bigcup L_{(j_1,\dots,j_d)}$.

Lemma 3.1. *(cf.* [EG2, Theorem 2.2]*) $G_{k'}$ acts freely on U_d.*

Proof. Since $G_{k'}$ is a closed subgroup of $B_{k'}$ and the open set U_d is $B_{k'}$ invariant, it suffices to show that $B_{k'}$ acts freely on U_d.

To do this we must show that the map $B_{k'} \times U_d \to U_d \times U_d$ given on k_s points by

$$(A, A_1, \dots, A_d) \mapsto (AA_1, \sigma_2(A\sigma_2^{-1}(A_2)), \dots, \sigma_d(A\sigma_d^{-1}(A_d))).$$

is a closed embedding.

First we show that the image Z of $B_{k'} \times U_d$ is closed in $U_d \times U_d$. Let $(A_1, A_2, \dots A_d, C_1, \dots, C_d)$ be matrix coordinates on $U_d \times U_d$. Expanding the inverse out in terms of the adjoint we see that the image is contained in the subvariety defined by the matrix equations

$$\sigma_j\sigma_i^{-1}(\det A_i)C_j = \sigma_j\sigma_i^{-1}(C_i \text{ Adj } A_i)A_j.$$

Suppose that a 2d-tuple of matrices $(A_1, A_2, \dots, A_d, C_1, C_2, \dots C_d) \in U_d \times U_d$ satisfies the matrix equations above. At least one of the A_i and one of the C_j is invertible because we are in $U_d \times U_d$. Let $A = \sigma_i^{-1}(C_iA_i^{-1})$. Substituting into our equations we see that $C_l = \sigma_l(A\sigma_l^{-1}(A_l))$ for all l. Moreover A is invertible since C_j is invertible and $C_j = \sigma_j(A\sigma_j^{-1}(A_j))$. Hence every point satisfying the matrix equations is in the image Z of $B_{k'} \times U_d$, so Z is closed.

The variety Z is covered by open sets of the form

$$\{(A_1, A_2, \dots A_j, \dots A_d, AA_1, \dots, \dots AA_j, \dots A_d) | \det A_j \neq 0\}.$$

These open sets are isomorphic to $V^{d-1} \times B_{k'}$, where V^{d-1} is the $d-1$-fold cartesian product of V. Hence the image is smooth, in particular, normal. The action of $G_{k'}$ on U_d is set-theoretically free so $G_{k'} \times U_d \to Z$ is a birational bijection. By Zariski's main theorem (cf. [Borel, AG18]) a birational bijection of a normal varieties is an isomorphism, so $G_{k'} \times U_d \to Z$ is an isomorphism. Therefore, $G_{k'} \times U_d \to U_d \times U_d$ is a closed embedding. $\square$

Remark. The proof of [EG2, Theorem 2.2] is incomplete; the last paragraph of the above argument is needed.

For any basis of V, there is a natural choice of basis so that with respect to this basis, if $g \in G(k_s)$, $\Phi(g)$ is represented by the block diagonal matrix

$$\begin{bmatrix} \rho(g) & & & & \\ & {}^{\sigma_2}\rho(g) & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & {}^{\sigma_d}\rho(g) \end{bmatrix}$$

This representation is not defined over k because the Galois group acts by permuting the blocks. More precisely, we have the following. Given a $d \times d$ matrix M, let $M[n]$ denote the $nd \times nd$ matrix whose ij block is $M_{ij} \cdot I_n$, where I_n is the $n \times n$ identity matrix. If $\sigma \in \mathrm{Gal}(k'/k)$, let J_σ denote the permutation matrix corresponding to the permutation $\sigma_i \mapsto \sigma\sigma_i$. In matrix form, for $g \in G(k_s)$,

$${}^{\sigma}\Phi(g) = J_\sigma[n]^{-1}\Phi(g)J_\sigma[n].$$

We will show that Φ is k'-isomorphic to a freely good representation defined over k. Choose a primitive element α for the extension $k' \supset k$, and let A be the $d \times d$ matrix with $A_{ij} = \sigma_j(\alpha^i)$. A is invertible since $\alpha, \sigma_2(\alpha), \ldots \sigma_d(\alpha)$ are exactly the roots of the irreducible polynomial $f \in k[x]$ of α over k, so $\det A = \prod_{i<j}(-1)^d(\sigma_i(\alpha) - \sigma_j(\alpha)) \neq 0$. The Galois group acts by $\sigma(A) = AJ_\sigma$. Consider the morphism $\Psi : G_{k'} \to \mathrm{GL}(V^{\oplus d})$ defined by by $\Psi(g) = A[n]\Phi(g)A[n]^{-1}$ for $g \in k_s$. Then ${}^{\sigma}\Psi(g) = \Psi(g)$ for any $\sigma \in \mathrm{Gal}(k'/k)$; hence Ψ is defined over k. Each of the subspaces $L_{(j_1,\ldots,j_d)}$ is $G_{k'}$-invariant under the action Ψ. Moreover, because each $L_{(j_1,\ldots,j_d)}$ is a vector subspace of $V^{\oplus d}$ defined over k, the corresponding sub-representations $G_{k'} \to \mathrm{GL}(L_{(j_1,\ldots,j_d)})$ are also defined over k. Therefore Ψ is obtained by base change from a freely good representation of G.

Case II. The general case In this case we may assume that there is a freely good k'-rational representation $\rho : G_{k'} \to \mathrm{GL}(V)$ defined over a finite normal extension k' of k. Then $k' \supset k$ factors as $k' \supset k'' \supset k$, with k'/k'' purely inseparable of degree p^n and k''/k Galois. The Frobenius endomorphism on V induces a group homomorphism of $\mathrm{GL}(V)$. Composing ρ with the n-th power of Frobenius on $\mathrm{GL}(V)$ we obtain a representation defined over k''. Because the Frobenius has finite kernel, this representation will no longer be faithful. However, the action of Frobenius is trivial on geometric points, so G will act properly on an open set whose complement is a union of linear subspaces. We can now use the Galois descent argument of Case I to obtain a good k-rational representation of G.

4. Characterization of solvable groups by good representations

In this section we show that if char $k \neq 2$, every reductive group with a good representation is a torus. However, many of the results of this section are valid in arbitrary characteristic, and we only need that char $k \neq 2$ in part of the proof of Theorem 1.1. We will explicitly say when we start assuming this; until then char k is arbitrary.

Let T be the diagonal torus in SL_2 and $N(T)$ the normalizer of T. We will first show that $N(T)$ has no good representations. We begin by recalling some facts about $N(T)$. Let

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

We will also write

$$H(t) = \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix}.$$

The group $N(T)$ is generated by T and J; it has two components, T and $J(T)$. The action of SL_2 on its two dimensional standard representation V induces an action on $S(V^*) \cong k[x,y]$, given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} : y \mapsto ay - cx, \qquad x \mapsto -by + dx.$$

Let W_i denote the subspace of $S(V^*)$ spanned by x^i and y^i; this is an irreducible representation of $N(T)$, of dimension 2 (if $i > 0$). Let W_0' denote the 1-dimensional irreducible representation of $N(T)$ on which T acts trivially and J acts by multiplication by -1.

If char $k \neq 2$, the group $N(T)$ is linearly reductive; that is, its action on any representation is completely reducible [GIT, p.191]. The next lemma shows that much of this survives in arbitrary characteristic.

Lemma 4.1. *Let V be a representation of $N(T)$.*

(1) *As a representation of $N(T)$, V splits as a direct sum of $N(T)$-submodules:*

$$V = V_0 \oplus \bigoplus_{i>0} V_{\pm i}.$$

Here $V_{\pm i}$ is the sum of the i and $-i$ weight spaces of T on V, and V_j is the j-weight space.

(2) *The action of $N(T)$ on $V_{\pm i}$ $(i > 0)$ is completely reducible, and $V_{\pm i}$ is isomorphic as $N(T)$-module to a direct sum of copies of W_i.*

(3) *If char $k \neq 2$, then V_0 is isomorphic to a direct sum of copies of W_0 and W_0'.*

Proof. (1) Because the action of T on V is completely reducible, we can decompose $V = \oplus V_i$ as T-module. As $JH(t)J^{-1} = H(t^{-1})$, we have $JV_i = V_{-i}$. Hence $V_{\pm i}$ is an $N(T)$-submodule and we get the desired direct sum decomposition of V.

(2) Let $v_1, \ldots, v_d$ be a basis for V_i ($i > 0$). The map $v_r \mapsto y$, $Jv_r \mapsto -x$ defines an isomorphism of the span of v_r, Jv_r (denoted $\langle v_r, Jv_r \rangle$) with W_i, and the map $W_i^{\oplus d} \to V_{\pm i}$, taking the r-th component to $\langle v_r, Jv_r \rangle$, is an $N(T)$-module isomorphism.

(3) Decompose the 0-weight space of V into the $+1$ and -1 eigenspaces of J; these are isomorphic to sums of copies of W_0 and W_0', respectively. $\square$

The proof of the following result was motivated by [GIT, Example 0.4].

Theorem 4.1. *The group $N(T)$ has no good representations.*

Proof. If a group G has good representations, then so does $G_{\bar{k}}$, so we may assume that k is algebraically closed. Suppose that V is a representation of $N(T)$, and let $U \subset V$ be the complement of a finite set of invariant linear subspaces $\mathcal{S}$. We will show that $N(T)$ does not act properly on $U = V - \cup_{L \in \mathcal{S}} L$. The strategy of the proof is as follows. Consider the action map $\Phi : N(T) \times U \to U \times U$. We will find a closed subvariety Z of $N(T) \times U$ whose closed points are of the form

$$\left(\begin{bmatrix} 0 & -\lambda^{-1} \\ \lambda & 0 \end{bmatrix}, v_\lambda \right)$$

whose image is not closed in $U \times U$. Hence Φ is not proper, so the representation is not good.

We now carry out the proof. Decompose $V = \oplus V_i$, where V_i is the i-weight space of V for T. Pick $u \in U$, and write $u = \sum u_i$ where $u_i \in V$. Some of the u_i may be 0; let d be the dimension of the space spanned by the nonzero u_i.

Step 1. If $a_i \neq 0$ for all i with $u_i \neq 0$, then $w = \sum a_i u_i \in U$. Indeed, suppose not; then $w \in L$ for some $L \in \mathcal{S}$. For almost all choices $t_1, \ldots, t_d$ of d elements of k^*, the vectors

$$H(t_q)w = \sum_p t_q^{i_p} a_{i_p} u_{i_p} \in L$$

are linearly independent. (Here $i_1, \ldots, i_d$ are the indices i_p with $u_{i_p} \neq 0$.) This follows because the $d \times d$ matrix A with entries

$$A_{pq} = t_q^{i_p}$$

is nonsingular for almost all $t_1, \dots, t_d$. (This is because $\det A$ is a sum of monomials, where each monomial is a product of one term from each row and each column. Each monomial has different multi-degree, so $\det A$ is not the zero polynomial.) Therefore, the vectors $H(t_q)w$ span the same space as the u_i, so $u \in L$, contradicting our assumption that $u \in U$. We conclude that $w \in U$, as claimed.

A similar argument shows that $Ju_0 + \sum_{i \neq 0} u_i \in U$.

Step 2. There exists an element $u' = \sum u'_i \in U$ with $Ju'_i = u'_{-i}$ for all $i > 0$. To see this, suppose $u_{-j} \neq Ju_j$ for some $j > 0$. Let $W_j \subset V_j \oplus V_{-j}$ be the subspace of vectors of the form $v_j + Jv_j$ ($v_j \in V_j$). Note that W_j generates $V_j \oplus V_{-j}$ as $N(T)$-module. Consider the affine linear subspace

$$B = \sum_{i \neq \pm j} u_i + W_j.$$

We claim that $B \cap U$ is nonempty. If it is empty, then because B is affine linear and is contained in a finite union of the subspaces in $\mathcal{S}$, we see that $B \subset L$ for some $L \in \mathcal{S}$. But then the span of B is contained in L, so $\sum_{i \neq \pm j} u_i \in L$ and $W_j \subset L$. Because L is $N(T)$-stable, $V_j \oplus V_{-j} \subset L$ as well; so

$$u \in \sum_{i \neq \pm j} u_i + (V_j \oplus V_{-j}) \subset L,$$

contradicting $u \in U$. We conclude that $B \cap U$ is nonempty. Replacing u by an element of $B \cap U$, which we again call u, we do not change u_i for $i \neq \pm j$, but we obtain $u_{-j} = Ju_j$. Iterating this process, we obtain u' of the desired form.

Replacing u by u', we will assume that $Ju_i = u_{-i}$ for all $i > 0$. From the $N(T)$-module isomorphism of $\langle u_i, Ju_i \rangle$ with $\langle x^i, y^i \rangle$, we see that for $i > 0$,

$$\begin{bmatrix} 0 & -\lambda^{-1} \\ \lambda & 0 \end{bmatrix} : u_i \mapsto \lambda^i u_{-i}, \qquad u_{-i} \mapsto (-1)^i \lambda^{-i} u_i.$$

Note also that

$$\begin{bmatrix} 0 & -\lambda^{-1} \\ \lambda & 0 \end{bmatrix} u_0 = Ju_0.$$

Step 3. Define

$$v_\lambda = u_0 + \sum_{i>0} (u_{-i} + \lambda^{-i} u_i)$$

$$v'_\lambda = Ju_0 + \sum_{i>0} (u_{-i} + (-1)^i \lambda^{-i} u_i).$$

For all $\lambda \neq 0$, both v_λ and v'_λ are in U (by Step 1). Define Z to be the closed subvariety of $N(T) \times U$ whose points are the pairs

$$(\begin{bmatrix} 0 & -\lambda^{-1} \\ \lambda & 0 \end{bmatrix}, v_\lambda).$$

Then

$$\Phi(Z) = \{(v_\lambda, v'_\lambda)\}_{\lambda \neq 0} \subset U \times U.$$

Consider the point

$$(v, v') = (u_0 + \sum_{i>0} u_{-i}, Ju_0 + \sum_{i>0} u_{-i}).$$

Reasoning as in Step 1 shows that u is in the $N(T)$-module generated by v or v', so if either v or v' were in L then u would be, but this is impossible as $u \in U$. Hence v and v' are in U, so $(v, v') \in U \times U$. Also, (v, v') is not in $\Phi(Z)$, but is in the closure of $\Phi(Z)$ in $U \times U$. We conclude that Φ is not proper, so the representation is not good. $\square$

4.1. **Proof of Theorem 1.1.** Let G be a connected nonsolvable linear algebraic group. Consider the surjective map $\pi : G \to G_1 = G/\mathcal{R}_u G$, where $\mathcal{R}_u G$ is the unipotent radical of G and G_1 is reductive. Because G is not solvable, G_1 is not trivial or a torus. Let T be a maximal torus of G. Then $T_1 = \pi(T)$ is a maximal torus of G_1, and π induces an isomorphism of Weyl groups $W(T, G) \to W(T_1, G_1)$ [Borel, 11.20]. (Here $W(T, G) = N_G(T)/Z_G(T)$ where N_G and Z_G denote normalizer and centralizer of T in G, and similarly for G_1.) Because $\ker \pi$ is a unipotent group, $\pi|_T : T \to T_1$ is an isomorphism. Note that $Z_G(T) = T \cdot (\mathcal{R}_u G)^T$ [Borel, 13.17]. Because G_1 is reductive, this fact (applied to G_1) implies that $Z_{G_1}(T_1) = T_1$. Moreover, any $g_1 \in N_{G_1}(T_1)$ can be lifted to $g \in N_G(T)$. This follows because the isomorphism of Weyl groups above, and the structure of the centralizers, imply that each component of $N_{G_1}(T_1)$ is the image of a surjective map of a component of $N_G(T)$.

As G_1 is not a torus, there is a root α and a homomorphism $\phi_\alpha : \mathrm{SL}_2 \to G_1$ with kernel either trivial, or the set of matrices $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ with $a^2 = 1$. Moreover (using the subscript SL_2 to denote terms for SL_2 defined in the previous subsection), $\phi_\alpha(T_{\mathrm{SL}_2}) \subset T$ and $J_1 := \phi_\alpha(J_{\mathrm{SL}_2}) \in N_{G_1}(T_1)$. (See [J, p.176] for these facts.) Let $H_1 = \phi_\alpha(N(T_{\mathrm{SL}_2}))$; its identity component $H_1^0 = \phi_\alpha(T_{\mathrm{SL}_2}) \subset T_1$. Because H_1 is a finite image of $N(T_{\mathrm{SL}_2})$, it has no good representations (and hence neither does G_1).

Up to this point, char k has been arbitrary; now we assume that char $k \neq 2$.

Because $\pi|_T$ is an isomorphism there is a unique subgroup $H^0 \subset T$ projecting isomorphically to H_1^0. As noted above, we can choose a lift $J \in N_G(T)$ of $J_1 \in N_{G_1}(T_1)$. Write $J = J_s J_u$ for the Jordan decomposition of J. Because char $k \neq 2$, J_1 is semisimple, so $\pi(J_u) = 1$. Therefore we can replace J by J_s and assume J is semisimple. Now, J^2 corresponds to the identity element in the Weyl group (as J_1^2 does), so $J^2 \in Z_G(T) = T \cdot (\mathcal{R}_u G)^T$. Since J is semisimple, we conclude that $J^2 \in T$. As J_1^2 is in the subgroup H_1^0 of T_1 and T maps isomorphically to T_1, we conclude that $J^2 \in H^0$. Therefore the group H generated by H^0 and J maps isomorphically to H_1, and thus has no good representations. Therefore G has no good representations. This proves Theorem 1.1.

The proof of Theorem 1.1 yields the following weaker statement in characteristic 2. Note that Levi decompositions need not exist in positive characteristic [Borel, 11.22].

Corollary 4.1. *Suppose char $k = 2$. If the connected algebraic group G has a Levi decomposition and if G has a good representation, then G is solvable. In particular, any connected reductive group with a good representation is diagonalizable.*

Proof. Suppose $G = LN$ where L is reductive and N unipotent. If G has a good representation then so does L. As proved above, this implies that L is a torus, so G is solvable. □

5. Proof of Theorem 1.2

If G has a representation V that contains an open set U whose complement is a finite union of invariant subspaces such that G acts with finite stabilizers on U, then $G_{\overline{k}}$ also has such a representation. Thus we can assume that k is algebraically closed.

Assume that G is not solvable and let V be a representation of G. Since the characteristic is 0, G has a Levi subgroup L. Since G is assumed to be non-solvable, L contains a Borel subgroup which is not a torus. Hence L contains a non-trivial unipotent subgroup N.

Since the characteristic is 0 and L is reductive, V decomposes as a direct sum $V = V_1 \oplus V_2 \dots V_p$ of irreducible L-modules. Every vector in the subspace $V^N = V_1^N \oplus V_2^N \oplus \dots \oplus V_p^N$ has positive dimensional stabilizer. Since N is unipotent, $V_i^N \neq 0$ for each i, so $L(V_i^N)$ spans all of V_i. Hence the subset $LV^N = L(V_1^N \oplus \dots \oplus V_p^N)$ which consists of vectors with positive dimensional stabilizers cannot be contained in any proper L-invariant subspace. Since L is a subgroup of G this means

that LV^N is not contained in any proper G-invariant subspace. Hence V does not have properties (1) and (2). $\square$

6. Examples and complements

In this section we discuss "set-theoretic" versions of the conditions freely good and good. We will say a representation V is set-theoretically freely good (resp. set-theoretically good) if it contains a nonempty open subset U whose complement is a union of invariant subspaces, such that G acts with trivial stabilizers (resp. finite stabilizers) on U (cf. Theorem 1.2). Surprisingly, these conditions are not enough to characterize solvability in arbitrary characteristic.

Example 6.1. Let V be the standard representation of SL_2 and let $V_d = S(V^*)$ be the vector space of homogeneous forms of degree d. As in Section 4, SL_2 acts on V_d. If $p = \operatorname{char} k$ is an odd prime then $W_p = V_{2p-2} \oplus V_1$ is a set-theoretically freely good representation of SL_2. The reason is as follows. The stabilizer of any pair of forms $(f(x,y), l(x,y))$ is trivial as long as $l(x,y) \neq 0$ and the coefficient of $x^{p-1}y^{p-1}$ in f is non-zero. Since the characteristic is p, the subspace $L_{2p-2} \subset V_{2p-2}$ of forms with no $x^{p-1}y^{p-1}$ term is an SL_2 invariant subspace (cf. [J, II2.16]). Thus, SL_2 acts with trivial stabilizers on the open set $U_p = W_p \smallsetminus (W_{2p-2} \oplus V_1 \cup V_{2p-2} \oplus 0)$.

In characteristic 2, the representation $W_2 = V_2 \oplus V_1$ is not set-theoretically freely good because the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ stabilizes the pair $(x^2y^2, x+y)$. However, W_2 is set-theoretically good.

In positive characteristic, we do not know if the group SL_n admits set-theoretically good representations for $n \geq 3$.

Example 6.2. Assume k is algebraically closed and $\operatorname{char} k \neq 2$. Then $G = PGL_2$ has no representation which is set-theoretically freely good. Indeed, let

$$g = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and let $H = \{1, g\}$. If V is any representation of G, V^H generates V as a representation of G. Indeed, this holds if V is irreducible, since for any vector v, the vector $v + gv$ is a nonzero H-invariant. Because $\operatorname{char} k \neq 2$, the action of H is completely reducible, so if

$$0 \to V_1 \to V_2 \to V_3 \to 0$$

is an exact sequence of G-modules, then the corresponding sequence of H-invariants is also exact. By induction, we may assume that V_1^H and

V_3^H generate V_1 and V_3 as G-modules, and a diagram-chase then shows that V_2^H generates V_2 as a G-module.

Hence if V is any representation, there is no proper invariant linear subspace of V containing V^H. Therefore V is not set-theoretically freely good.

A similar argument shows that PGL_n and GL_n do not have set-theoretically freely good representations.

We conclude with a proposition about the inductive construction of good representations.

Proposition 6.1. *Let G be a connected linear algebraic group and H a normal subgroup. Assume k is algebraically closed. If H and G/H have set-theoretically freely good representations, then so does G.*

Proof. For this proof only, we will use "good" to mean "set-theoretically freely good". Let W be a good representation of H, with M_i a finite set of proper invariant subspaces containing the vectors with nontrivial stabilizers. Because G/H is affine [Borel, Theorem 6.8], the vector bundle $G \times^H W$ is generated by a finite dimensional space of global sections Γ. We will view sections of the vector bundle as regular functions $\gamma : G \to W$ satisfying $\gamma(gh) = h^{-1} \cdot \gamma(g)$, where on the right side we are using the action of H on W. The action of G on the space of sections of the vector bundle corresponds to the left action of G on regular functions: $(g \cdot \gamma)(g_0) = \gamma(g^{-1}g_0)$. Because the action of G on regular functions is locally finite, by enlarging the space Γ, we may assume Γ is stable under the G-action.

Define L_i to be the subspace of Γ consisting of those elements of Γ which are sections of $G \times^H M_i$. Each L_i is a G-stable subspace of Γ. Let Γ^0 denote the complement of the L_i in Γ.

Let V be a good representation of G/H, viewed as a representation of G via the map $G \to G/H$. We claim that $V \oplus \Gamma$ is a set-theoretically good representation of G. Indeed, let V_j be a finite set of invariant subspaces of V containing the vectors with nontrivial stabilizer. It suffices to show that the vectors with nontrivial stabilizer in $V \oplus \Gamma$ are contained in the union of the subspaces $V_j \oplus \Gamma$ and $V \oplus L_i$. To see this, let (v, γ) be in the complement of these subspaces. so $v \notin V_j$ and $\gamma \notin L_i$ for any i, j. We must show that $\mathrm{stab}_G(v, \gamma)$ is trivial. First, $\mathrm{stab}_G(v, \gamma) \subset \mathrm{stab}_G(v) = H$. Let $h \in \mathrm{stab}_G(v, \gamma)$. As above, we will view γ as a function $G \to W$. Because γ is not in any L_i, we have γ is not a section of $G \times^H M_i$ for any i. In other words, the open subsets $\gamma^{-1}(W \setminus M_i)$ of G are nonempty. Choose g_0 in the intersection of these sets, so $s(g_0) \notin M_i$ for any i. Our hypothesis implies that $h \cdot \gamma = \gamma$.

By definition, we have

$$(h \cdot \gamma)(g_0) = \gamma(h^{-1} g_0) = \gamma(g_0(g_0^{-1} h^{-1} g_0)) = (g_0^{-1} h g_0)\gamma(g_0).$$

But $\operatorname{stab}_H \gamma(g_0) = \{1\}$, so we conclude $h = 1$ as desired. $\square$

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Department of Mathematics, University of Missouri, Columbia MO 65211

E-mail address: `edidin@math.missouri.edu`

Department of Mathematics, University of Georgia, Boyd Graduate Studies Research Center, Athens, GA 30602

E-mail address: `wag@math.uga.edu`